International Journal of Applied and Universal Research E- ISSN No: 2395-0269
Volume III, Issue 1, Jan-Feb. 2016 Available online at: www.ijaur.com

ON WILLEM DE SITTER COSMOLOGICAL MODEL OF THE UNIVERSE

R K Dubey[1], Anil Saini[2] Nilam Yadav[3]
1, 2&3 Department of Mathematics, Govt. Model Science College Rewa (MP)

***ABSTRACT**:* A de Sitter Universe is a Cosmological solution of Einstein's field equation of general Relativity which is named after Willem de Sitter. Also we have discussed in mathematics and physics, a de Sitter space or space time, of a sphere in ordinary Euclidean space and Penrose Diagram of De Sitter Space. Also we have discuss Pure de sitter space is the unique vacuum solution to the Einstein equation with maximal symmetry and constant positive curvature.

KEYWORDS: De Sitter Space, Willem se Sitter Cosmological Model.

INTRODUCTION

The Dutch astronomer Willem de Sitter (1872-1934) gave important contributions to the rise of relativistic Cosmology. The debate from 1916 to 1918 between de Sitter and Albert Einstein (1879-1955) is a fundamental chapter in the history of the scientific view of the universe. In fact, during such a debate both Einstein and de Sitter formulated their own mathematical expressions for the metric of the universe as a whole.

EINSTEIN STATIC MODEL OF THE UNIVERSE

In the fall of 1916 Einstein debated with de Sitter on the problem of suitable boundary conditions at infinity. According to the Principle of Relativity, Einstein tried to obtain values for the $g\mu\nu$'s at infinity that was invariant for all transformations. He avoided this difficulty by replacing such boundary conditions with the condition of closure, introducing a "finite and yet unbounded universe". Einstein proposed a spherical model of the universe, in which the matter was uniformly and homogenously distributed.

This static solution had line element:

$$ds^2 = dx_4^2 - g_{\alpha\beta}\, dx_\alpha dx_\beta; \qquad (1)$$

$$g_{\alpha\beta} = -\left[\delta_{\alpha\beta} + \frac{x_\alpha x_\beta}{R^2 - (x_1^2 + x_2^2 + x_3^2)}\right]. \qquad (2)$$

R was the radius of curvature of the three-space ($x1$, $x2$, $x3$), that was everywhere orthogonal to the time dimension $x4$.

This model fully achieved the relativity of inertia. There was not any independent property of space which claimed to the origin of inertia, so the latter was entirely produced by masses in the universe. The condition of spatial closure ensured that both the gravitational potential and the hypothetical average density of ponder able matter remained constant *in space*.

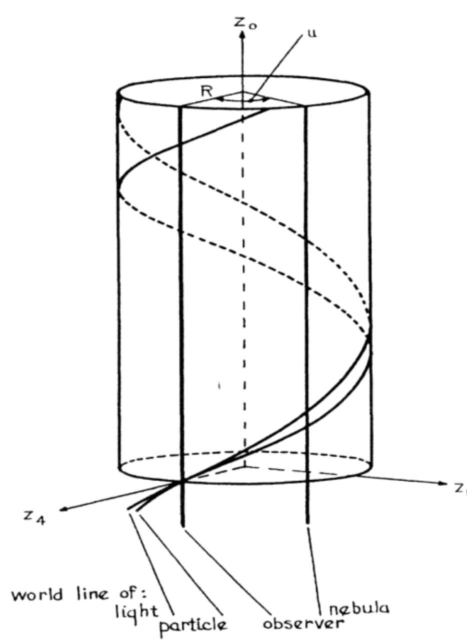

Figure 1: Einstein's "cylindrical" universe. One spatial dimension is disregarded. The vertical axis is along the direction of time [from *Robertson 1933*, p. 70].

Einstein modified his field equation accounting for the supposed static nature of the universe, i.e. to preserve the gravitational potential and the density of matter constant in time. He inserted the so-called *cosmological term*, namely the fundamental tensor $g\mu\nu$ multiplied by $-\lambda$, a universal but unknown constant:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \lambda g_{\mu\nu} = -kT_{\mu\nu} \qquad (3)$$

In this way field equations expressed the observational evidence of the static equilibrium of the universe. The new constant λ, the radius of the universe R, and the mean density of "world matter" ρ were strictly connected:

$$\lambda = \frac{kpc^2}{2} = \frac{1}{R^2} \qquad (4)$$

In Einstein model the metric of the universe could be given as a solution of relativistic field equations with the cosmological term. Both the general covariance and the laws of conservation of momentum and energy were still satisfied.

3 De Sitter "empty" model of the universe

Right after Einstein model appeared; de Sitter proposed his own solution of field equations. The Dutch astronomer admired Einstein conception of the universe "as a contradiction free chain of reasoning", and gave a different solution also maintaining the λ-term. However de Sitter preferred the original relativistic theory of gravitation, "without the undeterminable λ, which is just philosophically and not physically desirable".

3.1 The "mathematical postulate of relativity of inertia"

De Sitter approached the cosmological problem in a different way. It was mainly Paul Ehrenfest (1830-1933) who suggested him a mathematical conception of inertia, which led de Sitter to propose a finite and "empty" universe.

De Sitter proposed a distinction between the "world matter" and the "ordinary matter". The former was hypothetically distributed through space with density $\rho 0$. The latter corresponded to observable objects as planets and stars, i.e. to locally condensed world matter with density $\rho 1$. By this assumption, de Sitter pointed out that "inertia is produced by the whole of world matter, and gravitation by its local deviations from homogeneity".

Neglecting all pressures and internal forces, and supposing all matter to be at rest, the energy-momentum tensor became:

$$T_{44} = (p_o + p_1)c^2 g_{44} \,. \qquad (5)$$

De Sitter made the hypothesis to neglect gravitation on large-scale, and to take $\rho 0$ constant.

According to de Sitter, the three-dimensional finite world proposed by the Einstein said the "material relativity requirement", or equivalently the "material postulate of relativity of inertia" .

The Dutch astronomer pointed out that the relativistic field equations were "the fundamental ones": the postulate that at infinity all $g\mu\nu$'s were invariant for all transformations was more important than the "Machian" postulate of inertia introduced by Einstein. In fact in Einstein model, for the hypothetical value $R = \infty$, the whole of $g\mu\nu$'s degenerated to

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This set of values was invariant for all transformations for which, at infinity, $t0 = t$. In other words, in Einstein cylindrical world it was possible to find systems of reference in which the $g\mu\nu$'s only depended on the space-variables, and not on the "time". However the "time" of such a system had "a separate position", because it was "the same always and everywhere". For such a reason, according to de Sitter, the time coordinate in Einstein model was nothing else than an absolute time, and there the world matter took "the place of the absolute space in Newton's theory, or of the inertial system" . De Sitter proposed that the potentials should have degenerate at infinity to the values:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

According to him, "if at infinity all $g\mu\nu$'s were *zero*, then we could truly say that the whole of inertia, as well as gravitation, is thus produced. This is the reasoning which has led to the postulate that at infinity all $g\mu\nu$'s shall be zero". De Sitter called this requirement the "mathematical relativity condition", or the "mathematical postulate of relativity of inertia" .In fact, such a condition corresponded to the possibility that "the world as a whole can perform random motions without us (within the world) being able to observe it": "the postulate of the invariance of the $g\mu v$'s at infinity - de Sitter stated - has no real physical meaning. It is purely mathematical"

3.2 A universe without "world matter"

In a letter to Einstein de Sitter proposed his own solution of the metric of the universe as a whole, actually the second relativistic model in modern Cosmology. The Dutch astronomer considered field equations with the λ-term and without matter, i.e. by assuming $\rho 0 = 0$:

$$R_{\mu v} - \frac{1}{2} g_{\mu v} R - \lambda g_{\mu v} = 0. \qquad (6)$$

These equations could be satisfied by the $g\mu v$'s given by the metric:

$$ds^2 = \frac{-dx^2 - dy^2 - dz^2 + c^2 dt^2}{\left[1 - \frac{\lambda}{12}\left(c^2t^2 - x^2 - y^2 - z^2\right)\right]^2}. \qquad (7)$$

The coordinates (x, y, z, t) could have infinite values, on condition that $g\mu v$'s were null at infinity. Such a condition was equivalent to the finiteness of the world in natural (proper) measure. In fact the length of any semi-axis in natural measure was:

$$L_\alpha = \int_0^{\infty} \sqrt{-g_{\alpha\alpha}}\, dx_\alpha. \qquad (8)$$

A finite world (i.e. A finite value of $L\alpha$) necessary implied $g\alpha\alpha = 0$ for

$x\alpha \to \infty$, and vice versa .

De Sitter pointed out that in his model no world matter was necessary, and the insertion of the λ-term satisfied the mathematical postulate of relativity of inertia. In this system there was not any universal time, nor any difference between the "time" and the other coordinates: none of these coordinates had any physical meaning. The cosmological constant determined the value of the curvature radius R:

$$\lambda = \frac{3}{R^2} \qquad (9)$$

By using an imaginary "time"-coordinate $\xi 4 = ict$, the geometry of de Sitter world was that of a 4-dimensional hyper sphere which could be described in a 5-dimensional Euclidean space:

$$R^2 = \xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 + \xi_5^2. \qquad (10)$$

In hyper-spherical coordinates the metric of such a *four-dimensional world*

Resulted:

$$ds^2 = -R^2\{dw^2 + sin^2 w[d\zeta^2 + sin^2\zeta(d\psi^2 + sin^2\psi d\theta^2)]\} \qquad (11)$$

where $0 \leq \theta \leq 2\pi$; $0 \leq \psi, \zeta, \omega \leq \pi$. Equivalently, by replacing the imaginary "time"-coordinate $\xi 4$ with a real time-coordinate ($\xi 4 \to i\xi 4$), the geometry of de Sitter world corresponded to a 4-dimensional hyperboloid in a 4+1- dimensional Minkowski space-time:

$$R^1 = \xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 + \xi_5^2. \qquad (12)$$

By pseudo-spherical coordinates (with $i\omega 0 = \omega$), the metric of space-time resulted:

$$ds^2 = -R^2\{d\acute{\omega}^2 + sinh^2\acute{\omega}[d\zeta^2 + sin^2\zeta(d\psi^2 + sin^2\psi d\theta^2)]\} \qquad (13)$$

Where $0 \leq \theta \leq 2\pi$; $0 \leq \psi, \zeta \leq \pi$; $-\infty < \omega 0 < +\infty$.

The potentials in the hyper spherical coordinate system were:

$$g_{\mu v} = -\left[\delta_{\mu v} + \frac{x_\mu x_v}{R^2 - (x_1^2 + x_2^2 + x_3^2 + x_4^2)}\right]. \qquad (14)$$

Thus the metric proposed by de Sitter,

$$ds^2 = \frac{-dx^2 - dy^2 - dz^2 + c^2 dz^2}{\left[1 - \frac{\lambda}{12}(c^2t^2 - x^2 - y^2 - z^2)\right]^{2}}, \qquad (15)$$

Could be obtained by the stereographic projection of the 4-dimensional hyper- sphere to the Euclidean space, or equivalently by the projection of the hyperboloid to a 3+1-dimensional Minkowski space-time.

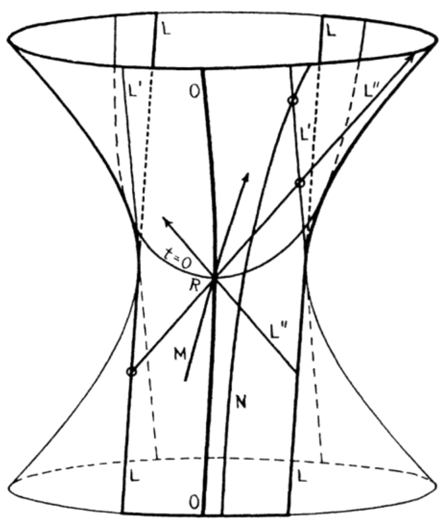

Figure 3: The de Sitter universe. Two spatial dimensions are suppressed [From *Robertson 1933*, p. 71]

"If a single test particle - de Sitter wrote to Einstein - existed in the world, that is, there were *no* sun and stars, etc., it would have inertia". Essentially, in the universe proposed by de Sitter a suitable metric was obtained without any "physical" masses. Such forms of matter as stars and nebulæ were to be regarded as "test particles" in a fixed background metric, which curvature was determined by the cosmological constant.

3.2.1 Einstein criticism

Einstein acknowledged de Sitter's solution to be "very interesting" , but "must have been disappointed", and tried to discard this anti Machian solution: "I cannot grant - Einstein wrote to de Sitter - your solution any physical possibility". In fact, the cosmological term took a fundamental role in de Sitter model in order to involve a sort of spatial (and not material) origin of inertia. "The g$\mu\nu$ field - Einstein

Replied to de Sitter - should *be fully determined by matter, and not be able to exist without the latter"*.

$$1 - \frac{\lambda}{12}(c^2t^2 - x^2 - y^2 + z^2) = 0 \qquad (16)$$

At first Einstein objected that the hyperboloid surface was a singularity. On this surface there was a discontinuity, because the $g44$ term "jumped" from $+\infty$ to $-\infty$, and $g\alpha\alpha$'s from $-\infty$ to $+\infty$. Such a surface lied in the physically finite, but it was not possible to assume infinite values for the potentials, because of the supposed static nature of the universe and the small velocities measured on stars. Moreover, the four-dimensional continuum proposed by de Sitter did not have the property that all its points were equivalent. In fact it had a preferred point, i.e. the center of the conic section

$$1 + (c^2t^2 - x^2 - y^2 - z^2) = 0 \qquad (17)$$

De Sitter replied that the hyper-surface involved a finite natural spatial distance and an infinite natural temporal distance. Thus the discontinuity was only apparent, and this problem was "not interesting". Also the supposed preferred point was later shown to be a geometrical consequence of that choice of coordinates, and not a true physical aspect. "My four-dimensional world - de Sitter remarked to Einstein - also has the λ-term, but no *world matter".*

3.2.2 Elliptical geometry

In order better to compare his own model with Einstein solution, de Sitter proposed another expression of the metric. By using spherical polar coordinates, he represented the hyperboloid universe (system *B*) as the Einstein universe (system A), i.e. As 3-dimensional hyper-spheres embedded in a 4-dimensional Euclidean space:

$$ds_A^2 = -dr^2 - R^2 sin^2 \frac{T}{R}(d\psi^2 + sin^2\psi d\theta^2) + c^2dt^2 , \qquad (18)$$

$$ds_B^2 = -dr^2 - R^2 sin^2 \frac{r}{R} (d\psi^2 + sin^2 \psi d\theta^2) + cos^2 \frac{r}{R} c^2 dt^2. \quad (19)$$

De Sitter pointed out that, between the possible forms of space with constant curvature, the elliptical space was more preferable than the spherical one. In the elliptical space, which also was closed respect to its dimensions, any two straight lines could not have more than one point in common. Einstein agreed with de Sitter on the choice of elliptical space, but he noticed that the spherical geometry he used in the Cosmological *Considerations in the General Theory of Relativity only* was an approximation. According to Einstein, it served to show "through an idealization, that a spatially closed (finite) system is possible. The system could actually be quite irregularly curved, also on a large scale, that is, it could relate to the spherical world like a potato's surface to a sphere's surface".

By the new expression of the line element (the so called *static form*) it was clear that all the points in de Sitter world were equivalent . However, as Einstein pointed out, however he still believed that this "anti-Machian" universe was not a physical possibility. The *g*44 coefficient of the temporal term in system *B depended* on position. In fact, being *g*44 = $cos^2 \frac{\pi}{R}$ such a potential changed its value from 1 (for *r* = 0) to 0 (for $r = \frac{\pi R}{2}$). According to Einstein, time clocks slowed down approaching $r = \frac{\pi R}{2}$: this null value for potential involved that all masses had the tendency to aggregate at this "equator" . "It seems Einstein wrote in 1918 - that no choice of coordinates can remove this discontinuity. We have to assume that de Sitter solution has a genuine singularity on the surface $r = \frac{\pi R}{2}$.

The de Sitter system does not look at all like a world free of matter, but rather like a world whose matter is concentrated entirely on the surface $r = \frac{\pi R}{2}$. According to Einstein, a *free of matter* solution of field equations was inconceivable. Through his critical comment to de Sitter solution, Einstein noticed that the cosmological constant did not involve any sort of spatial origin of inertia.

De sitter acknowledged Einstein remark to be correct, but gave a different interpretation. According to the Dutch astronomer, such a remark involved a philosophical, and not a physical requirement. In fact, the "equator" at $r = \frac{\pi R}{2}$ *was* at a finite distance in space, but was physically inaccessible. The velocity of a material particle became zero for $r = \frac{\pi R}{2}$. Thus a material particle which was on the polar line on the origin could have no velocity, nor energy. "All these results de Sitter stated sound very strange and paradoxical. They are of course, all due to the fact that *g*44 becomes zero for $r = \frac{\pi R}{2}$. We can say that on the polar line the four dimensional space is reduces to the three-dimensional space: *there is no time*, and consequently no motion". The time needed by a ray of light, or by a material particle, to travel by any point to the equator was infinite. Thus the singularity at $r = \frac{\pi R}{2}$ could never affect any physical experiment .The issue was solved by Felix Klein (1849-1925). In some correspondence with Einstein, the authoritative mathematician showed that the singularity at the equator in de Sitter universe could be eliminated by using the first coordinates of the hyperboloid form. Thus such a singularity could "simply be transformed away": it only was a geometrical consequence of the choice of coordinates. The matter-free model proposed by de Sitter was free of singularities, and its space-time points were all equivalent. At the end Einstein admitted that de Sitter solution existed.

4 CONCLUSION

"At the present time - de Sitter wrote in 1920 - the choice between the systems *A* and *B is* purely a matter of taste. There is no physical criterion as yet available to decide between them". However de Sitter noticed that these systems differed in their physical consequences. In fact, in de Sitter world a particle at rest would not have remained at rest unless it was at the origin. This mass test would have escaped far away because of the presence of the cosmological constant. Thus the de Sitter system to all appearances was static, and required a positive radial velocity for distant objects. This effect of recession was known as "de Sitter effect". At that time, and during the Twenties, this effect appeared to be connected in some manner with the first red-shift observations of many nebulæ. The interest in de Sitter effect survived until 1930, when truly non-static theoretical models of the universe were proposed to explain the red-shift problem and the astronomical evidences of a cosmic recession.

REFERENCES :

1. Bernstein, J. and Feinberg, G. (Eds.) *Cosmological constants: papers in modern cosmology*. New York: Columbia University Press. (1986)
2. M. J. Klein, A. J. Kox, J. Renn and R. Schulmann (Eds.) The collected papers of Albert Einstein - Vol. 4, the Swiss years: writings, *1912-1914* - English translation. Princeton: Princeton University Press. (1995)
3. R. Schulmann, A. J. Kox, M. Janssen and J. Illy (Eds.) The collected papers of Albert Einstein - Vol. 8, the Berlin years: correspondence, *1914-1918* - English translation. Princeton: Princeton University Press. (1998)
4. M. Janssen, R. Schulmann, J. Illy, C. Lenher and D. K. Buch- Wald (Eds.) The collected papers of Albert Einstein - Vol. 7, the Berlin years: writings, *1918-1921* - English translation. Princeton: Princeton University Press. (2002)
5. On Einstein's theory of gravitation, and its astronomical consequences. Monthly Notices of the Royal Astronomical Society, *76, 699-728*. (1916a)
6. On the relativity of rotation in Einstein's theory. Koninklijke Akademie van Wetenschappen. Section of Science, *19, 527-532*. (1916b)
7. On Einstein's theory of gravitation, and its astronomical consequences. Second paper. Monthly Notices of the Royal Astronomical Society, *77, 155-184*. (1916)
8. De Sitter, W. on the relativity of inertia. Re-marks concerning Einstein's latest hypothesis. Koninklijke Akademie van Wetenschappen. Section of Science, *19, 1217-1224*. (1917a)
9. De Sitter, W. On Einstein's theory of gravitation, and its astronomical consequences. Third paper. Monthly Notices of the Royal Astronomical Society, *78, 3-28*. (1917b)
10. De Sitter, W. On the curvature of Space. Koninklijke Akademie van Wetenschappen. Section of Science, *20, 229-243*. (1917c)
11. De Sitter, W. Further remarks on the solutions of the field equations of Einstein's theory of gravitation. Koninklijke Akademie van Wetenschappen. Section of Science, *20 II, 1309-1312*. (1918)
12. De Sitter, W. On the possibility of statistical equilibrium of the universe. Koninklijke Akademie van Wetenschappen. Section of Science, *23, 866-868*. (1920)
13. Einstein, A. Physikalische grundlagen einer gravitations theorie. Naturforschende Gesellschaft in Zu¨rich. Vierteljahrss- chrift, *58, 284-290*. Engl. tr.: Physical foundations of a theory of gravitation. In [*CPAE 1995]*, doc. 16 (pp. 192-197). (1913)
14. Einstein, A. *U*¨ber die spezielle und allgemeine relativit¨ats theorie (gemeinverst¨and-lich*)*. Braunschweig: Vieweg. Engl. tr.: On the special and the general theory of relativity. A popular account. In [*CPAE 1996]*, doc. 42 (pp. 247-420). (1917a)
15. Einstein, A. Kosmologische betrachtungen zur all- gemeinen relativit¨atstheorie. K¨oniglich Preuβische Akademie der Wissenschaften (Berlin). Sitzungsberichte, *142*-152. Engl. tr.: Cosmological considerations in the general theory of relativity. In [*CPAE 1996]*, doc. 43 (pp. 421-432). (1917b)
16. Einstein, A. Prinzipielles zur allgemeinen relaivit¨atstheorie. Annalen der Physik*, 55, 241-244*. Engl. tr.: On the foundations of the general theory of relativity. In [*CPAE 2002]*, doc. 4 (pp. 33-35). (1918a)
17. Einstein, A. Kritisches zu einer von Hrn. De Sitter gegebenen l¨osung der gravitations gleichungen. K¨oniglich Preuβische Akademie der Wissenschaften (Berlin). Sitzungsberichte, *270-272*. Engl. tr.: Critical comment on a solution of the gravitational field equations given by Mr. de Sitter. In [*CPAE 2002]*, doc. 5 (pp. 36-38). (1918b)
18. Einstein, A. Geometrie underfahrung. Berlin: Springer. Engl. tr.: Geometry and experience. In [*CPAE 2002]*, doc. 52 (pp. 208-222). (1921)
19. Janssen, M. The Einstein - de Sitter debate and its after math [(http://www.tc.umn.edu/](http://www.tc.umn.edu/)~janss011). (2003)
20. Kerszberg, P. The invented universe: the Einstein- De Sitter controversy (1916-17) and the rise of relativistic cosmology. Oxford: Clarendon Press. (1989)
21. North, J. D. The measure of the universe: a history of modern cosmology. Oxford: Clarendon Press. (1965)
22. Pais, A. 'Subtle is the Lord...' - The science and the life of Albert Einstein. Oxford: Oxford University Press. (1982)
23. Robertson, H. P. Relativistic Cosmology. Reviews of Modern Physics, *5, 62-88*. (1933)